Engineering Notes

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Linearization of the Boundary-Layer Equations of the Minimum Time-to-Climb Problem

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Introduction

IN using singular perturbation techniques to obtain numerical solutions of nonlinear optimal control problems, the most difficult and costly part of the computation often is to determine suitable solutions of the zeroth-order boundary-layer equations (ZOBLE's). The purpose of these equations is to model the nonlinear dynamical behavior of the "fast" variables. (The higher order equations are linear and thus, in principle, pose no computational problems.) If there are m fast state variables, then the ZOBLE's are of order $2m$. The zeroth-order outer solution evaluated at the boundary is a stationary point of these equations and a solution must be found which decays exponentially to this stationary point and satisfies m boundary conditions. Such a solution is called here a boundary-layer solution.

The known results[1,2] give two essential requirements (in addition to some technical restrictions) that are sufficient for the existence of boundary-layer solutions. The first is an eigenvalue condition ensuring a form of local stationary-point stability; the second is a restriction that the initial conditions be in the domain of influence of the stationary point. In practice, it is often possible to check the first condition but rarely possible to check the second. Usually, the best that can be done is a local analysis of the stability properties, accomplished by linearizing the equations about the outer solution.

Aiken and Lapidus[3] have suggested solving the linearized ZOBLE's as an approximation of the nonlinear equations. This is justified on the basis that the boundary-layer solution decays with a large exponential. This procedure would not generally seem to be justified, however, because the purpose of the ZOBLE's is, in fact, to model the nonlinear behavior of the fast variables on a time scale stretched in proportion to the rate of decay. Thus the accuracy of the linear solution will depend on the degree of system nonlinearity and the "distance" between the boundary conditions and the zeroth-order outer solution.

Nevertheless, the savings in computational effort with this approximation are substantial and this approach should be investigated in applications. One use of the linear solution is to provide a means of predicting the m unknown initial conditions for the nonlinear ZOBLE's.

In Ref. 4, the two-point boundary-value problem arising from a general optimal control problem is formally linearized and the known stability properties of such a linear system are reviewed. This Note applies these results to the minimum time-to-climb (MTC) problem and derives and solves the linearized ZOBLE's for this problem. A numerical example is presented.

Analysis

As derived in Refs. 5 and 6, the initial ZOBLE's of the MTC problem have the following form when linearized about the outer solution:

$$\frac{d\alpha}{d\tau} = v\beta \qquad \frac{d\beta}{d\tau} = c\alpha + \frac{1}{v\lambda_\gamma}\delta$$
$$\frac{d\theta}{d\tau} = e\alpha - c\delta \qquad \frac{d\delta}{d\tau} = -\frac{\lambda_\gamma}{v}\beta - v\theta \tag{1}$$

where

$$c = \frac{f}{v} \qquad e = -\frac{\lambda_\gamma f^2}{v} - \lambda_E p_{hh} \qquad f = \frac{2}{v^2} - \frac{B_h}{B} \tag{2}$$

In Eqs. (1), α, β, θ, and δ are linear perturbations in altitude, flight path angle, the adjoint variable associated with altitude, and the adjoint variable associated with flight path angle, respectively. The coefficients in Eqs. (1) are evaluated on the zeroth-order outer solution at $t=0$, and the independent variable is defined by $\tau = t/\epsilon$. Additional nomenclature and algebraic details, nonessential for the present Note, are found in Refs. 5 and 6.

The structure of the coefficient matrix of Eqs. (1) and the pattern of its eigenvalues are discussed in Ref. 4. In particular, there will generally be two eigenvalues of the form

$$S_1 = -a + ib \qquad S_2 = -a - ib \tag{3}$$

with a and b positive real numbers. The general solution of Eqs. (1) which decays exponentially is then

$$\alpha = A^{(1)} e^{S_1\tau} + A^{(2)} e^{S_2\tau}$$
$$\beta = \frac{S_1}{v} A^{(1)} e^{S_1\tau} + \frac{S_2}{v} A^{(1)} e^{S_1\tau}$$
$$\theta = \left[\frac{e}{S_1} - \frac{c\lambda_\gamma}{S_1}(S_1^2 - vc)\right] A^{(1)} e^{S_1\tau} + \left[\frac{e}{S_2} - \frac{c\lambda_\gamma}{S_2}(S_2^2 - vc)\right] A^{(2)} e^{S_2\tau}$$
$$\delta = \lambda_\gamma (S_1^2 - vc) A^{(1)} e^{S_1\tau} + \lambda_\gamma (S_2^2 - vc) \cdot A^{(2)} e^{S_2\tau} \tag{4}$$

where $A^{(1)}$ and $A^{(2)}$ are constants to be determined by the initial conditions. The initial conditions on Eqs. (4) are the differences between the unperturbed initial conditions and the values on the outer solution:

$$\alpha(0) = h_0 - h = \Delta_h \quad \text{and} \quad \beta(0) = \gamma_0$$
(problem A, initial flight path angle fixed) (5)

$$\alpha(0) = h_0 - h = \Delta_h \quad \text{and} \quad \delta(0) = -\lambda_\gamma$$
(problem B, initial flight path angle free) (6)

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Where h_0 and γ_0 are the initial conditions on altitude and flight path angle.

First consider problem A. From Eqs. (4) and (5), $A^{(1)}$ and $A^{(2)}$ are determined as

$$A^{(1)} = \frac{v\gamma_0 - S_2\Delta_h}{S_1 - S_2} \qquad A^{(2)} = \frac{-v\gamma_0 + S_1\Delta_h}{S_1 - S_2} \tag{7}$$

Putting these in Eqs. (4) and using Eqs. (3) yields

$$\alpha(\tau) = e^{-a\tau}\left\{\Delta_h \cos b\tau + [v\gamma_0 + a\Delta_h]\frac{\sin b\tau}{b}\right\}$$

$$\beta(\tau) = e^{-a\tau}\left\{\gamma_0 \cos b\tau - [av\gamma_0 + (a^2+b^2)\Delta_h]\frac{\sin b\tau}{bv}\right\} \tag{8}$$

For problem B, a similar procedure gives

$$\alpha(\tau) = \frac{\Delta_h e^{-a\tau}}{2ab}\left[2ab\cos b\tau + \left(\frac{1}{\Delta_h} - f + a^2 - b^2\right)\sin b\tau\right]$$

$$\beta(\tau) = \frac{\Delta_h e^{-a\tau}}{2v}\left[\frac{1}{a}\left(\frac{1}{\Delta_h} - f - a^2 - b^2\right)\cos b\tau + \frac{1}{b}\left(-\frac{1}{\Delta_h} + f - a^2 - b^2\right)\sin b\tau\right] \tag{9}$$

The second of these predicts the value of γ_0 when it is left free:

$$\gamma_0 = \beta(0) = \frac{1 - \Delta_h(f + a^2 + b^2)}{2av} \tag{10}$$

The solutions of the linearized terminal boundary-layer equations are similar to those of the initial layer. To obtain the terminal solutions from Eqs. (8) and (9), change $\tau = t/\epsilon$ to $\tau' = (T-t)/\epsilon$, v to $-v$, h_0 to h_f, and evaluate coefficients at the terminal values of the zeroth-order outer solution.

Use of the linearized zeroth-order boundary-layer solutions has two major advantages over dealing with the nonlinear equations. First, there is no need for the repetitive, coupled, numerical integrations needed to solve a nonlinear two-point boundary-value problem. Second, the integrals on the interval $[0, \infty)$ of the zeroth-order boundary-layer variables which are needed to obtain first-order terms are available in closed form without recourse to numerical integrations to arbitrary cutoff points. The question remains, however, as to the numerical validity of the linear solution (investigated in the next section).

A possible use of the linear solution is to predict the unknown initial conditions for the nonlinear equations. From Eqs. (4) and (7), the value of the unknown initial condition in problem A is predicted by the linear solution to be

$$\lambda_{\gamma_0} = \lambda_\gamma [1 - 2av\gamma_0 - \Delta_h(a^2 + b^2 + f)] \tag{11}$$

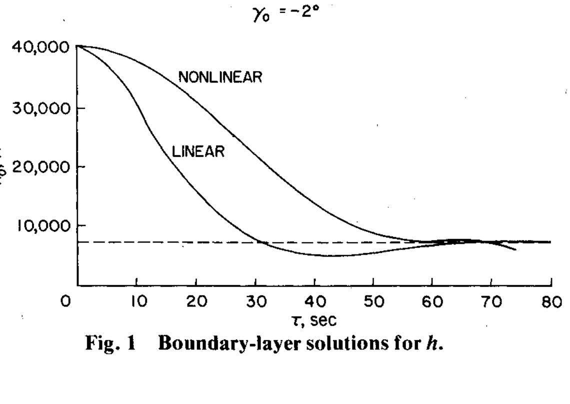

Fig. 1 Boundary-layer solutions for h.

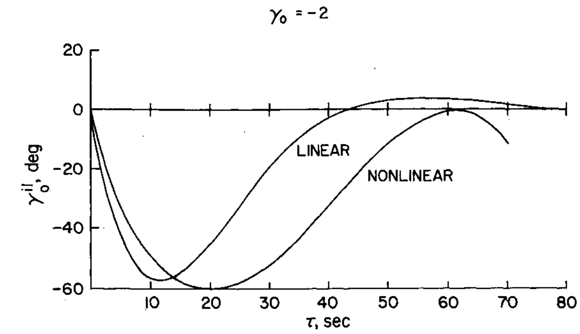

Fig. 2 Boundary-layer solutions for γ.

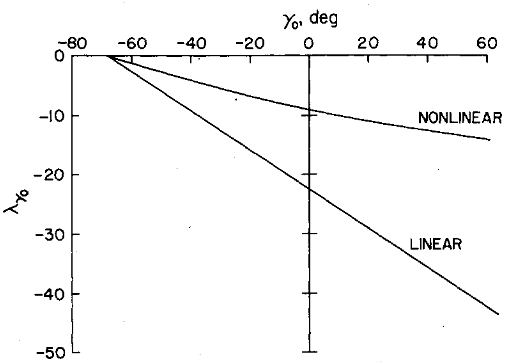

Fig. 3 Prediction of $\lambda_\gamma(0)$.

For γ_0 free (problem B), $\lambda_{\gamma_0} = 0$ and, for this case, Eq. (11) necessarily gives the same value for γ_0 as does Eq. (10). The numerical validity of Eq. (11) is also investigated in the following section.

Numerical Example

Consider the same example used in Refs. 5 and 6. It is desired to find the minimum time to climb between a Mach number of 0.5 and an altitude of 40,000 ft and a Mach number of 2.0 and an altitude of 80,000 ft for a certain hypothetical high-performance aircraft. It was determined in Ref. 5 that, for the initial boundar layer, $a = 0.0625$ and $b = 0.0738$.

The linear solution for the zeroth-order initial boundary layer for a representative problem A ($\gamma_0 = -2$ deg) is shown in Figs. 1 and 2. The nonlinear solution is also shown for comparison. The two solutions show fairly good qualitative agreement in both h_0^{il} and γ_0^{il} in that time constants, periods, and initial slopes are about the same for the linear and nonlinear solutions. In particular, the steep dive to $\gamma \approx 60$ deg is predicted by both solutions. Quantitatively, however, the agreement is poor. For some values of τ, there is considerable difference in the linear and nonlinear values of h_0^{il} and γ_0^{il}. Further, it is apparent that the integrals of functions of h_0^{il} needed to obtain the first-order solutions will differ considerably. Thus it seems that the linear solutions are of doubtful value in a numerical solution. This is due, of course, to the high degree of nonlinearity in the problem.

Figure 3 compares the values of the unknown initial condition λ_{γ_0} as a function of specified γ_0. The linear curve is obtained from Eq. (11). Obtaining the nonlinear curve is quite laborious since, for each value of γ_0, the nonlinear ZOBLE's must be repetitively solved for different values of λ_{γ_0} until satisfactory boundary-layer solutions are obtained. Considering the case with which the linear prediction is obtained and the lack of any other good estimating procedures, the agreement evidenced in Fig. 3 is sufficient to make the linear prediction of λ_{γ_0} useful as a starting value in the integration of the nonlinear ZOBLE's.

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Valve Delay Effects on Bending Limit Cycles of a Deadband System

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REFERENCE 1† contains an analysis of the interactions of a deadband control system with a flexible spacecraft. A simple analytic criterion is developed which determines whether or not it is possible to obtain unstable bending limit cycles—i.e. limit cycles in which both sets of nozzles fire at approximately the bending frequency and in such phase as to expend control gas rapidly for the nonuseful purpose of maintaining a large bending amplitude. This criterion was verified by digital simulation computations.

Of the several simplifying idealizations made in Ref. 1, it is believed that the assumption of equal on and off solenoid valve delays introduces the greatest differences between the analysis predictions and the behavior of a real system. The analysis of this Note therefore supplements Ref. 1 by replacing the assumption of equal on and off delays with more realistic assumptions. Figure 1 is derived from test data on a representative solenoid valve which was subjected to electrical commands of varying duration. The chamber pressure was measured in a small plenum downstream of the valve and upstream of the nozzle. The data records show 1) time of applied voltage on, 2) time of voltage off, 3) pressure history vs time. The latter exhibits a delay after voltage on, then a sloped rise to a level plateau pressure which holds for a time after voltage off, then falls along a slope to zero. The time difference between voltage on and off is called C and is plotted on the horizontal axis of Fig. 1 as the "Voltage Command Interval." Let the plateau pressure be P, the total integral of pressure vs time be I, and the integral after voltage off be I_{off}. The square points on Fig. 1 are (I_{off}/P) and the circle points are $C-[(I-I_{off})/P]$. Using the idealization of

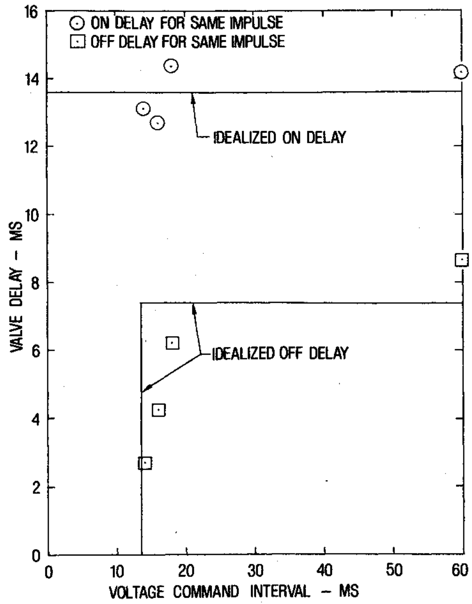

Fig. 1 Test of solenoid valve.

instantaneous rise and fall, the points represent the equivalent delays which result in the same total impulse and same impulse after off command as the real valve exhibits. It is seen that the assumption of a constant on delay is reasonably in accord with the test data. The assumption of constant off delay for commands longer than ~15 ms is less accurate but still fairly reasonable. The assumption that the on delay is equal to the off delay is much less reasonable. The graphical estimates drawn on Fig. 1 give a value of ~13.6 ms for on but only ~7.4 ms for off delay. Thus the assumption of equal on and off delays will result in thrust pulses that are always $(13.6-7.4)=6.2$ ms too long; and any average value which is chosen for the equal on or off delay cannot represent the true value of on delay for which no thrust occurs. A much more accurate representation is obtained by defining the on and off delays as:

$$t_1 = t_{on} = 13.6 \text{ ms} = \text{constant}$$
$$t_3 = t_{off} = 0 \text{ for voltage command interval} \le t_{on}$$
$$t_3 = t_{off} = 7.4 \text{ ms for voltage command interval} > t_{on}$$

The use of these assumptions results in thrust pulses of approximately the true length and impulse for command lengths greater than t_{on}; but for command lengths less than or equal to t_{on} no thrust occurs, which is approximately in accord with the test data. Evidently a much superior analytic representation for the valve is obtained by allowing on and off delays to be unequal.

The analysis of Ref. 1 is unchanged through Eq. (19); Eq. (20), however, must be replaced by

$$\Delta\psi_{on} = \tan^{-1}\omega_1 t_2 + \phi + \omega_1 t_1 \tag{20a rev}$$

$$\Delta\psi_{off} = \tan^{-1}\omega_1 t_2 + \phi + \omega_1 t_3 \tag{20b rev}$$

Equations (21a) and (21c) are still valid. The work/energy Eq. (26) derivation is identical except that $\Delta\psi$ in all lower integration limits is replaced by $\Delta\psi_{on}$ and in all upper limits by $\Delta\psi_{off}$. Then a revised Eq. (43) is obtained from Eq. (26) by the same assumption. The results of the nozzle work integration are conveniently expressed in terms of the average and the difference of the valve on and off delays. Thus with

$$\Delta\psi \equiv \tan^{-1}\omega_1 t_2 + \phi + \tfrac{1}{2}\omega_1 (t_1 + t_3) \tag{20c}$$

$$S_1 \equiv \sin[\tfrac{1}{2}\omega_1 (t_1 - t_3)] \tag{20d}$$

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†Equations in this note are numbered to correspond to the equations in Ref. 1.